Summer semester 2021

BORDISMS AND TFTS - EXERCISE 2

(1) Framings

(a) Find at least 3 different framings on the strip $(0,1) \times (0,1)$, restricting to the below framings in a neighborhood of $\{0\} \times (0,1)$ and $\{1\} \times (0,1)$.



- (b) Which ones determine the same (which) orientation? How would you define "homotopy of framing"?
- (c) Can S^2 be framed?

Definition. Let B^n denote the *n*-dimensional ball as a manifold with boundary and S^n the *n*-dimensional sphere.

Given a 2-dimensional manifold M, we attach a *j*-handle $H^j \coloneqq B^j \times B^{2-j}$, for $j \in \{0, 1, 2\}$ via and a smooth embedding $f: S^{j-1} \times B^{2-j} \hookrightarrow \partial M$ as follows:

 $M \cup_f H^j \coloneqq \left(M \sqcup (B^j \times B^{2-j}) \right) / \sim$

where for $(p, x) \in S^{j-1} \times B^{2-j} \subset B^j \times B^{2-j}$, we set $f(p, x) \sim (p, x)$.

- (2) Attaching handles
 - (a) Find a smooth structure on $M \cup_f H^j$.
 - (b) Which surface do you obtain from attaching a 1-handle to a disk?
 - (c) Which surface do you obtain from attaching two 1-handles to a disk, i.e. from attaching an additional 1-handle to the surface obtained in part a?
 - (d) Build the torus by successively attaching handles to a disk.
- (3) Properties of the connected sum of manifolds
 - (a) Given *n*-manifolds M, M', and M'', show that the connected sum satisfies the following properties.

(i)	$M \# S^n \cong M,$	(neutral element)
(ii)	$M \# M' \cong M' \# M$, and	(commutativity)
(iii)	$(M \# M') \# M'' \cong M \# (M' \# M'').$	(associativity)

- (b) If M and M' are smooth *n*-manifolds, construct a smooth structure on the connected sum M # M'. Note that this is not unique, but defines a well-defined diffeomorphism class. You may like to read more details using isotopies in Chapter 8, Section 2 in Hirsch, Differential Topology¹.
- (4) Below is a list of several proofs of the classification theorem of 1-dimensional manifolds, using different tools. Read through one (or several) of them.
 - https://pnp.mathematik.uni-stuttgart.de/igt/eiserm/lehre/2014/Topologie/ Gale%20-%201-manifolds.pdf
 - http://www.math.boun.edu.tr/instructors/wdgillam/1manifolds.pdf
 - Appendix of https://www.maths.ed.ac.uk/~v1ranick/papers/milnortop.pdf, starting at p.55.

¹Can e.g. be accessed at https://www.researchgate.net/publication/268035774_Differential_ Topology.