summer semester 2020

Claudia Scheimbauer
Tashi Walde

Advanced Topics in Algebraic Topology — Exercise Sheet 13

Exercise class: Friday, 24th of July, 11-12

Website with further material, including exercise sheets:
`https://www.groups.ma.tum.de/algebra/scheimbauer/advanced-topics-in-algebraic-topology/`

The goal for this exercise sheet is to prove deRham's theorem relating singular cohomology of a manifold with its deRham cohomology. In particular this shows that deRham cohomology is a purely topological invariant.

This is essentially a summary of Chapter 18 in Lee's book (Introduction to smooth manifolds) where you can find the details.

We start by recalling the definition of the singular (co)homology (with coefficents in $\mathbb{R}$)[1] of a topological space X:

- For each $n \in \mathbb{N}$, consider the topological n-simplex

$$\Delta^n := \{x \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i \leq 1 \text{ and } \forall i \colon x_i \geq 0\} \tag{1}$$

  which is equipped with coface maps $d^i = d^i_n \colon \Delta^{n-1} \to \Delta^n$, induced by the inclusion $\mathbb{R}^n \to \mathbb{R}^{n+1}$ which inserts a zero at position $0 \leq i \leq n$.

- We define the singular chain complex $(\mathrm{C}_*(X, \mathbb{R}), \mathrm{d}_{\mathrm{sing}})$ of X as follows: The term $\mathrm{C}_n(X, \mathbb{R})$ is the $\mathbb{R}$-vector space with a basis given by the set of continuous maps $\Delta^n \to X$ (this vector space is *huge*); the differential $\mathrm{d}_{\mathrm{sing}} \colon \mathrm{C}_n(X, \mathbb{R}) \to \mathrm{C}_{n-1}(X, \mathbb{R})$ is given by sending a basis vector $\sigma \colon \Delta^n \to X$ to the alternating sum

$$\mathrm{d}_{\mathrm{sing}}(\sigma) := \sum_{i=0}^{n} (-1)^i \sigma \circ d^i_n \in \mathrm{C}_{n-1}(X, \mathbb{R}) \tag{2}$$

- The singular cochain complex $\mathrm{C}^*(X, \mathbb{R}) := \mathrm{C}_*(X, \mathbb{R})^\vee$ is obtained from $\mathrm{C}_*(X, \mathbb{R})$ by passing to the $\mathbb{R}$-dual vector space[2] in each term (this makes the huge vector space even more humongous).

- The singular homology $\mathrm{H}^{\mathrm{sing}}_*(X)$ is defined as the homology of the chain complex $\mathrm{C}_*(X, \mathbb{R})$; similarly the singular cohomology is defined as the cohomology of the cochain complex $\mathrm{C}^*(X, \mathbb{R})$. Note that in fact we have $\mathrm{H}^*_{\mathrm{sing}}(X) = \mathrm{H}^{\mathrm{sing}}_*(X)^\vee$; you may take this as the definition.

When X is a manifold, we can consider the subcomplex $\mathrm{C}^\infty_*(X, \mathbb{R})$ spanned by the *smooth* maps[3] $\Delta^* \to X$. By a suitable local smoothing procedure[4] which is a bit too technical to discuss here,

[1] Of course singular (co)homology is defined for more general coefficients, but we won't need that here

[2] Recall that the $\mathbb{R}$ dual of an $\mathbb{R}$-vector space V is the vector space $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$.

[3] The simplex Δ^n is not a smooth manifold, but a *smooth manifold with corners* which locally looks like $\mathbb{R}^k \times [0, \infty)^{k-n}$. Basically everything you learned this semester about manifolds with boundary carries over to such manifolds with corners.

[4] See Lee, Theorem 18.7

one can show that the inclusion $C_*^\infty(X,\mathbb{R}) \hookrightarrow C_*(X,\mathbb{R})$ is actually a homotopy equivalence of chain complexes. Hence we can redefine $H_*^{\text{sing}}(X)$ as the homology of $C_*^\infty(X,\mathbb{R})$; the dual of $H_*^{\text{sing}}(X)$ is then again $H^*_{\text{sing}}(X)$; in other words you may assume that all maps $\sigma\colon \Delta^n \to X$ are smooth.

In the following exercises you will show that $H^*_{\text{sing}}(X)$ is isomorphic to $H^*(X)$ by an explicit integration procedure.

Exercise 1. Show that there is a well defined map $R_X\colon H^*(X) \to H_*^{\text{sing}}(X)^\vee = H^*_{\text{sing}}(X)$, called the **deRham map of** X which sends (the class of) an n-form ω to (the class of) the linear functional $\int_- \omega\colon H_*^{\text{sing}}(X) \to \mathbb{R}$ given by $\sigma \mapsto \int_{\Delta^n} \sigma^*(\omega)$. Prove that R_X is natural in X.

Recall (or learn) that singular homology admits a Mayer-Vietoris exact sequence

$$\xrightarrow{\partial} H_n^{\text{sing}}(U\cap V) \to H_n^{\text{sing}}(U) \oplus H_n^{\text{sing}}(V) \to H_n^{\text{sing}}(X) \xrightarrow{\partial} H_{n-1}^{\text{sing}}(U \cap V) \tag{3}$$

whenever we cover X by two opens. See Lee, Theorem 18.4 for an explicit description of the maps in the sequence. Passing to duals then yields a Mayer-Vietoris sequence for singular cohomology.

Exercise 2. Prove that the deRham map $R_X\colon H^*(X) \to H_*^{\text{sing}}(X)$ is compatible with the Mayer-Vietoris sequences of $H^*(X)$ and $H_*^{\text{sing}}(X)$.

Exercise 3. Show that the deRham map of $\mathbb{R}^n$ is an isomorphism for all n.

Exercise 4. Use an inductive procedure to show that the deRham map is an isomorphism for each manifold X which admits a finite good cover.

One can in fact prove that the de Rham map of *every* manifold is an isomorphism (see Lee, Theorem 18.14).