

ADVANCED TOPICS IN ALGEBRAIC TOPOLOGY — EXERCISE SHEET 2

Exercise class: Friday, 8th May; 10-12

Website with further material, including exercise sheets:

<https://www.groups.ma.tum.de/algebra/scheimbauer/advanced-topics-in-algebraic-topology/>

Recall that a map $f: X \rightarrow \mathbb{R}$ (on a topological space) is called **locally constant** if each point $x \in X$ has a neighborhood on which f is constant.

Exercise 1. Let $U \subset \mathbb{R}^n$ be an open subset.

- (1) Prove that a continuous map $f: U \rightarrow \mathbb{R}$ is locally constant if and only if it is constant on each connected component of U .
- (2) Given two smooth functions $f, g: U \rightarrow \mathbb{R}$, show that $df = dg$ if and only if $f - g$ is locally constant.

Recall that a 1-form ω is called **closed** if $d\omega = 0$ and **exact** if there exists a function f such that $df = \omega$.

Exercise 2. Consider the circle $S^1 := \{z : |z| = 1\} \subset \mathbb{C}$ with the standard charts

$$\phi_+^{-1}: (0, 2\pi) \xrightarrow{\cong} S^1 \setminus \{1\}$$

$$\phi_-^{-1}: (-\pi, \pi) \xrightarrow{\cong} S^1 \setminus \{-1\}$$

given by $x \mapsto e^{ix}$. Show that $\phi_+^*(dx) \in \Omega^1(S^1 \setminus \{1\})$ and $\phi_-^*(dx) \in \Omega^1(S^1 \setminus \{-1\})$ glue to a well defined 1-form ω on S^1 . Is ω closed? Is ω exact?

Exercise 3. Consider the following 1-form on $\mathbb{R}^2 \setminus \{0\}$:

$$\omega := \frac{-ydx + xdy}{x^2 + y^2} \tag{1}$$

- (1) Show that ω is closed on $\mathbb{R}^2 \setminus \{0\}$.
- (2) On which of the following open subsets $\mathbb{R}^2 \setminus \{0\}$ is ω exact?
 - (a) All of $\mathbb{R}^2 \setminus \{0\}$.
 - (b) The open annulus $\{(x, y) : 1 < x^2 + y^2 < 2\}$.
 - (c) The upper half plane $\{(x, y) : y > 0\}$.
 - (d) The right half plane $\{(x, y) : x > 0\}$.
 - (e) The lower half plane $\{(x, y) : y < 0\}$.
 - (f) The complement of the negative x -axis: $\mathbb{R}^2 \setminus \{(x, 0) : x < 0\}$.

Hint: Consider the function $\theta: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ given by $re^{i\theta} \mapsto \theta$.

Exercise 4. Determine whether the 1-form

$$\frac{x\mathrm{d}x + y\mathrm{d}y}{(x^2 + y^2)^2}$$

on $\mathbb{R}^2 \setminus \{0\}$ is exact.

Exercise 5. Let $U \subset \mathbb{R}^2$ be an open disk of some radius r around some center (x, y) . Prove that every closed 1-form on U is exact.