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Advanced Topics in Algebraic Topology — Exercise Sheet 8

Exercise class: Friday, 19th of June, 11-12

Website with further material, including exercise sheets:
`https://www.groups.ma.tum.de/algebra/scheimbauer/advanced-topics-in-algebraic-topology/`

Exercise 1. (a) Use the Künneth formula to compute the cohomology of $T^n := \overbrace{\mathrm{S}^1 \times \cdots \times \mathrm{S}^1}^{n \text{ times}}$. Give an explicit basis!

(b) What is the Euler characteristic (as defined on the last sheet) of T^n?

(c) Determine the Poincaré dual $\eta_S \in \mathrm{H}^1(T^2)$ of the submanifold $S := \mathrm{S}^1 \times \{1\} \subset T^2$.

Definition 1 (Real and complex projective lines)**.** The **complex projective line** is the topological space

$$\mathbb{CP}^1 := \frac{\mathbb{C}^2 \setminus \{0\}}{\sim}$$

(with the quotient topology), where $v \sim w$ if and only if v and w span the same vector subspace $\langle v\rangle_{\mathbb{C}} = \langle w\rangle_{\mathbb{C}}$ of $\mathbb{C}^2$.

Replacing $\mathbb{C}$ by $\mathbb{R}$ yields the **real projective line** $\mathbb{RP}^1$.

Exercise 2. (a) Equip $\mathbb{RP}^1$ with the structure of a smooth manifold diffeomorphic to S^1.

(b) Equip $\mathbb{CP}^1$ with the structure of a smooth manifold which is diffeomorphic to the 2-sphere S^2.

Exercise 3 (Hopf fibration)**.** (a) Show that the projection $p\colon \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$ is a non-trivial fiber bundle with fibers diffeomorphic to $\mathbb{C}^\times$.

(b) Show that p contains a non-trivial fiber subbundle $\mathrm{S}^3 \to \mathrm{S}^2 \cong \mathbb{CP}^1$ with fibers diffeomorphic to S^1.

Exercise 4 (Tautological bundle on projective lines)**.** (1) Construct a non-trivial real vector bundle of rank 1 on $\mathbb{RP}^1$ such that the fiber over each point $[v] \in \mathbb{RP}^1$ is the $\mathbb{R}$-vector space $\langle v\rangle_{\mathbb{R}} \subset \mathbb{R}^2$. Have you seen this bundle before?

(2) Construct a non-trivial complex vector bundle of rank 1 on $\mathbb{CP}^1$ such that the fiber over each point $[v] \in \mathbb{CP}^1$ is the $\mathbb{C}$-vector space $\langle v\rangle_{\mathbb{C}} \subset \mathbb{C}^2$. How does this bundle relate to the one constructed in Exercise 3?

Exercise 5 (Bonus challenge)**.** Let M be a compact non-orientable manifold of dimension n. Prove that $\mathrm{H}^n(M) = 0$. (Hint: Consider the orientation double cover from sheet 5.)