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Algebraic Topology – Exercise 11

Definition. A sequence of groups $\{G_i\}_{i\in I}$ and group homomorphisms $\{f_i\}_{i\in I}$

$$\ldots \longrightarrow G_{i+1} \xrightarrow{f_{i+1}} G_i \xrightarrow{f_i} G_{i-1} \longrightarrow \ldots$$

is called *exact* if the image of each group homomorphism is equal to the kernel of the next, i.e. $\mathrm{im}(f_{i+1}) = \ker(f_i)$ for each $i \in I$[1].

(1) (a) If $0 \to A \xrightarrow{\alpha} B$ is an exact sequence, what can you say about α? Similarly, if you have an exact sequence of the form $A \xrightarrow{\beta} B \to 0$, what can you say about β?

(b) How many exact sequences of abelian groups of the form

$$0 \xleftarrow{f_0} \mathbb{Z}/2\mathbb{Z} \xleftarrow{f_1} \mathbb{Z}/4\mathbb{Z} \xleftarrow{f_2} \mathbb{Z}/4\mathbb{Z} \xleftarrow{f_3} \ldots$$

exist?

(c) Is there a short exact sequence of abelian groups of the form

$$0 \longleftarrow \mathbb{Z}/4\mathbb{Z} \longleftarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \longleftarrow \mathbb{Z}/4\mathbb{Z} \longleftarrow 0?$$

(2) (a) Let X and Y be topological spaces. Is every chain map $g : C_*^{\mathrm{sing}}(X) \to C_*^{\mathrm{sing}}(Y)$ induced by a map of topological spaces?

(b) Let $\mathbb{D}_*^n$ be the chain complex whose only non-trivial entries are in degrees n and $n-1$, with $\mathbb{D}_n^n = \mathbb{Z} = \mathbb{D}_{n-1}^n$. Its only non-trivial boundary operator is the identity:

$$\mathbb{D}_*^n := \big(.. \longleftarrow 0 \longleftarrow \mathbb{Z} \xleftarrow{\mathrm{id}} \mathbb{Z} \longleftarrow 0 \longleftarrow \ldots\big)$$

Similarly, let $\mathbb{S}_*^m$ be the chain complex whose only non-trivial entry is $\mathbb{S}_m^m = \mathbb{Z}$, i.e.

$$\mathbb{S}_*^m := \big(.. \longleftarrow 0 \longleftarrow \mathbb{Z} \longleftarrow 0 \longleftarrow \ldots\big)$$

Are there chain maps between $\mathbb{D}_*^n$ and $\mathbb{S}_*^m$?

(c) Let (C_*, ∂) and (C'_*, ∂') be two arbitrary chain complexes and let $f_* : C_* \to C'_*$ be a chain map. Assume that f_n is a monomorphism for all n. Do we then know that the maps $H_n(f_*)$ induced on homology are also monomorphisms?

[1] Note that the indexing set I can be either finite or infinite.

(3) Prove the *five lemma*: Consider a commutative diagram of abelian groups as below, where both rows are exact. Show that if α, β, δ, and ε are isomorphisms, then γ is also an isomorphism.

$$\begin{array}{ccccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \varepsilon} \\
A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E'
\end{array}$$

Definition. Let (D_*, ∂) be a chain complex. Denote by $[d] \in H_n^{\text{sing}}(D)$ the equivalence class of a cycle $d \in \ker(\partial_n)$. If $d, d_0 \in C_*^{\text{sing}}(D)$ are such that $d - d_0$ is a boundary, then d is said to be *homologous* to d_0.

(4) Let X be a path-connected, non-empty topological space and let $\omega : [0,1] \to X$ be a continuous path with $\omega(0) = x$ and $\omega(1) = y$. Recall from Exercise 1 on Sheet 9 that $\Delta^1 := \{(t_0, t_1) \in \mathbb{R}^2 : \sum_{i=0}^{1} t_i = 1, t_i \geqslant 0\}$. Define a singular 1-simplex $\alpha_\omega : \Delta^1 \to X$ as $\alpha_\omega(t_0, t_1) = \omega(1 - t_0)$. In other words, we have associated to a continuous path ω in X a 1-simplex α_ω on X. Let $\omega, \omega_1, \omega_2$ be paths in X. Using the above identification show that:

(a) Constant paths c_y at y in X are homologous to 0, i.e. the difference $\alpha_{c_y} - 0$ is the boundary of some 2-simplex.

(b) If $\omega_1(1) = \omega_2(0)$, we can define the concatenation $\omega_1 * \omega_2$. Then $\alpha_{\omega_1 * \omega_2} - \alpha_{\omega_1} - \alpha_{\omega_2}$ is a boundary.

(c) If $\omega_1(0) = \omega_2(0), \omega_1(1) = \omega_2(1)$ and if ω_1 is homotopic to ω_2 relative to $\{0, 1\}$, then α_{ω_1} and α_{ω_2} are homologous as singular 1-chains.

(d) Any 1-chain formed from a path of the form $\bar{\omega} * \omega$ is a boundary. Here $\bar{\omega}(t) := \omega(1 - t)$.

Proposition 1. For any non-empty path-connected topological space X there is an isomorphism $\pi_1(X, x)_{\text{ab}} \cong H_1^{\text{Sing}}(X)$.

(5) Guided proof of Proposition 1.

(a) Let $h : \pi_1(X, x) \to H_1^{\text{sing}}(X)$ be the map that sends the homotopy class $[\omega]_{\pi_1}$ of a closed path ω to its homology class $[\alpha_\omega] = [\omega]_{H_1}$. This is called the *Hurewicz homomorphism.* Show that h is well-defined and a group homomorphism. *Hint:* Use the previous exercise.

(b) Recall (or look up) the universal property of the abelianization of a group. Use it to construct a group homomorphism $h_{\text{ab}} : \pi_1(X, x)_{\text{ab}} \to H_1^{\text{sing}}(X)$.

(c) Construct an inverse to h_{ab} as follows: Choose, for any point $y \in X$, a path u_y from the base point x to y (for the base point x itself choose u_x to be the constant path). Let α be an arbitrary singular 1-simplex and let $y_i := \alpha(e_i)$, where e_i is the ith unit basis vector of $\mathbb{R}^2$. Define

$$\widetilde{\phi} : C_1^{\text{sing}}(X) \to \pi_1(X, x)_{\text{ab}}$$

on the generator α to be the class of the closed path $\widetilde{\phi}(\alpha) = [u_{y_0} * \alpha * \bar{u}_{y_1}]$, and extend linearly. Show that $\widetilde{\phi}$ is trivial on boundaries. *Hint:* Keep in mind that you map into something abelian.

(d) Conclude that $\widetilde{\phi}$ descends to a homomorphism $\phi : H_1(X) \to \pi_1(X, x)_{\mathrm{ab}}$. Show that it indeed is the inverse of h_{ab}.

(6) What computational results for $H_1(X)$ follows from the isomorphism in Proposition 1? Consider e.g. familiar topological spaces and different products of topological spaces.