MATRIX GENERATORS FOR THE REE GROUPS $^2G_2(q)$

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For the purposes of [K] and [KM] it became necessary to have 7×7 matrix generators for a Sylow-3-subgroup of the Ree groups $^2G_2(q)$ and its normalizer. For example in [K] we used the matrices produced here to show that in a seven dimensional representation the Jordan canonical form of any element of order nine is a single Jordan block of size 7. In [KM] we develop group recognition algorithms where at some stage we need to identify a certain subset of group elements with the Sylow-3-subgroup of $^2G_2(q)$. This identification is most easily done using the faithful matrix representation of small dimension given below. The process of producing low dimensional matrix representations for classical groups in defining characteristic is generally well understood. For exceptional groups of Lie type this does not appear to be true. For example on page 247, Carter [C] writes that we lack a matrix representation of conveniently small degree for the groups $G_2(q)$; the untwisted exceptional groups of lowest possible rank. The twisted exceptional groups of BN-pair rank 1 are the groups of type $^2B_2(q)$ and $^2G_2(q)$. Explicit matrix generators for the groups of type $^2B_2(q)$ are given in Chapter XI, § 3 of [BH], leaving the groups $^2G_2(q)$ as the only groups of BN-pair rank 1 for which there does not exist an explict matrix repesentation in defining characteristic in the literature. Here we give matrix generators for two distinct Sylow-3-subgroups of $^2G_2(q)$, thereby providing generators for the whole group. Starting with the Steinberg generators for a seven dimensional representation of $G_2(q)$ we construct our matrices following Carter [C], chapters 12 and 13. The matrices for the Steinberg generators of $G_2(q)$ were computed with the help of a computer program developed by the second author [L].

For our setup we let $G = G_2(K)$, where K is the algebraic closure of a finite field of characteristic 3. Let F a Frobenius endomorphism of G whose set of fixed points G^F is a Ree group of type $^2G_2(q)$, $q = 3^{2m+1}$. Let T be an F-invariant maximal torus of G, and let B and B^- be F-invariant Borel subgroups intersecting in T with unipotent radicals U respectively U^-. By N we denote the normalizer $N_G(T)$. Let Φ be the root system of G with respect to T and $\{a, b\}$ its base given by B, where a is a short and b a long root. Now U respectively U^- is generated by subgroups X_r respectively X_{-r}, where $r \in \Phi^+$ (the set of positive roots). The groups X_r are isomorphic to the additive group of the field K. We denote the elements of X_r by $X_r(t)$ where $t \in K$.

The reductive group G has an irreducible 7-dimensional representation over K (with highest weight $(1, 0)$) which can be found as follows: In characteristic 3, the 14-dimensional adjoint representation V of G has a 7-dimensional irreducible submodule. This submodule is spanned by those elements of the Chevalley basis of V which are labeled by short roots. The restriction of this representation to $^2G_2(q)$ remains irreducible.

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The second author has implemented a computer program which computes for an arbitrary Chevalley group explicit matrices for the root elements $X_r(t)$ in its adjoint representation with respect to a Chevalley basis. Using this for type G_2, we obtain the images of the $X_r(t)$ in the 7-dimensional representation with high weight $(1, 0)$ by cutting out the appropriate 7×7-blocks of the $X_r(t)$. (By abuse of notation we also denote these images by $X_r(t)$ in the sequel.)

These programs use the computer algebra packages GAP [GAP] and CHEVIE [GHLMP]. They work along the construction of the Chevalley groups as explained in Carters book [C]. It is planned to make them available to interested users as part of a larger package [L] for computing characters and highest weight representations of reductive groups.

The following list gives the matrices $X_r(t)$ for G. The reader can check that they satisfy the Steinberg relations [C] 12.2.1, where the structure constants are chosen as in the table on page 211 of [C]. (We denote zero entries by dots.)

$$X_a(t) = \begin{pmatrix} 1 & t & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ . & . & 1 & t & 2t^2 & . & . \\ . & . & . & 1 & t & . & . \\ . & . & . & . & 1 & . & . \\ . & . & . & . & . & 1 & 2t \\ . & . & . & . & . & . & 1 \end{pmatrix}, \qquad X_{-a}(t) = \begin{pmatrix} 1 & . & . & . & . & . & . \\ t & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & 2t & 1 & . & . & . \\ . & . & 2t^2 & 2t & 1 & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & 2t & 1 \end{pmatrix},$$

$$X_b(t) = \begin{pmatrix} 1 & . & . & . & . & . & . \\ . & 1 & t & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & 2t & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{pmatrix}, \qquad X_{-b}(t) = \begin{pmatrix} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ . & t & 1 & . & . & . & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ . & . & . & . & 2t & 1 & . \\ . & . & . & . & . & . & 1 \end{pmatrix},$$

$$X_{a+b}(t) = \begin{pmatrix} 1 & . & 2t & . & . & . & . \\ . & 1 & . & t & . & 2t^2 & . \\ . & . & 1 & . & . & . & . \\ . & . & . & 1 & . & t & . \\ . & . & . & . & 1 & . & t \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{pmatrix}, \qquad X_{-a-b}(t) = \begin{pmatrix} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ 2t & . & 1 & . & . & . & . \\ . & 2t & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ . & 2t^2 & . & 2t & . & 1 & . \\ . & . & . & . & t & . & 1 \end{pmatrix},$$

$$X_{2a+b}(t) = \begin{pmatrix} 1 & . & . & 2t & . & . & 2t^2 \\ . & 1 & . & . & 2t & . & . \\ . & . & 1 & . & . & t & . \\ . & . & . & 1 & . & . & 2t \\ . & . & . & . & 1 & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{pmatrix}, \qquad X_{-2a-b}(t) = \begin{pmatrix} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ t & . & . & 1 & . & . & . \\ . & 2t & . & . & 1 & . & . \\ . & . & t & . & . & 1 & . \\ 2t^2 & . & . & t & . & . & 1 \end{pmatrix},$$

$$X_{3a+b}(t) = \begin{pmatrix} 1 & . & . & . & 2t & . & . \\ . & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & t \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \end{pmatrix}, \qquad X_{-3a-b}(t) = \begin{pmatrix} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & . & 1 & . & . & . \\ 2t & . & . & . & 1 & . & . \\ . & . & . & . & . & 1 & . \\ . & . & t & . & . & . & 1 \end{pmatrix},$$

$$X_{3a+2b}(t)=\begin{pmatrix}1&.&.&.&.&2t&.\\.&1&.&.&.&.&t\\.&.&1&.&.&.&.\\.&.&.&1&.&.&.\\.&.&.&.&1&.&.\\.&.&.&.&.&1&.\\.&.&.&.&.&.&1\end{pmatrix},\quad X_{-3a-2b}(t)=\begin{pmatrix}1&.&.&.&.&.&.\\.&1&.&.&.&.&.\\.&.&1&.&.&.&.\\.&.&.&1&.&.&.\\.&.&.&.&1&.&.\\2t&.&.&.&.&1&.\\.&t&.&.&.&.&1\end{pmatrix}.$$

With our choice of structure constants the automorphism F has a particularly nice form. Let π be the involution on Φ which permutes $\pm a$ and $\pm b$, $\pm(a+b)$ and $\pm(3a+b)$, and $\pm(2a+b)$ and $\pm(3a+2b)$. Let $\theta=3^m$. Now

$$F(X_r(t))=X_{\pi(r)}(t^{\lambda(\pi(r))\theta}),$$

where $\lambda(\pi(r))$ is 1 if $\pi(r)$ is short and 3 if $\pi(r)$ is long, see for example Proposition 12.4.1 of [C] and the discussion on page 225.

Now every element of U is of the form

$$X_a(t_1)X_b(t_2)X_{a+b}(t_3)X_{2a+b}(t_4)X_{3a+b}(t_5)X_{3a+2b}(t_6)$$

with unique elements $t_1,\dots,t_6\in K$. Following the proof of [C] Proposition 13.6.3(vii) we confirm, using the computer program Maple, that

$$\begin{aligned}&F(X_a(t_1)X_b(t_2)X_{a+b}(t_3)X_{2a+b}(t_4)X_{3a+b}(t_5)X_{3a+2b}(t_6))=\\&\quad X_b(t_1^{3\theta})X_a(t_2^{\theta})X_{3a+b}(t_3^{3\theta})X_{3a+2b}(t_4^{3\theta})X_{a+b}(t_5^{\theta})X_{2a+b}(t_6^{\theta})=X_a(t_2^{\theta})X_b(t_1^{3\theta})\cdot\\&\quad\ X_{a+b}(t_1^{3\theta}t_2^{\theta}+t_5^{\theta})X_{2a+b}(t_1^{3\theta}t_2^{2\theta}+t_6^{\theta})X_{3a+b}(-(t_1t_2)^{3\theta}+t_3^{3\theta})X_{3a+2b}(-t_1^{6\theta}t_2^{3\theta}+t_4^{3\theta}).\end{aligned}$$

Comparing the coefficients of the factors and using that for $x\in K$ we have $x^{3\theta^2}=x^q=x$ iff $x\in F_q$, we get the parametrization of the F-stable elements of U.

Set $t=t_2$, $u=t_5$ and $v=t_6$. Then $U^F=\{x_S(t,u,v)\mid t,u,v\in F_q\}$, where

$$x_S(t,u,v)=\begin{pmatrix}1&t^{\theta}&-u^{\theta}&(tu)^{\theta}-v^{\theta}&f_1(t,u,v)&f_2(t,u,v)&f_3(t,u,v)\\.&1&t&u^{\theta}+t^{\theta+1}&-t^{2\theta+1}-v^{\theta}&f_4(t,u,v)&f_5(t,u,v)\\.&.&1&t^{\theta}&-t^{2\theta}&v^{\theta}+(tu)^{\theta}&f_6(t,u,v)\\.&.&.&1&t^{\theta}&u^{\theta}&(tu)^{\theta}-v^{\theta}\\.&.&.&.&1&-t&u^{\theta}+t^{\theta+1}\\.&.&.&.&.&1&-t^{\theta}\\.&.&.&.&.&.&1\end{pmatrix}$$

with

$$\begin{aligned}\theta&=3^m,\\f_1(t,u,v)&=-u-t^{3\theta+1}-(tv)^{\theta},\\f_2(t,u,v)&=-v-(uv)^{\theta}-t^{3\theta+2}-t^{\theta}u^{2\theta},\\f_3(t,u,v)&=t^{\theta}v-u^{\theta+1}+t^{4\theta+2}-v^{2\theta}-t^{3\theta+1}u^{\theta}-(tuv)^{\theta},\\f_4(t,u,v)&=-u^{2\theta}+t^{\theta+1}u^{\theta}+tv^{\theta},\\f_5(t,u,v)&=v+tu-t^{2\theta+1}u^{\theta}-(uv)^{\theta}-t^{3\theta+2}-t^{\theta+1}v^{\theta},\\f_6(t,u,v)&=u+t^{3\theta+1}-(tv)^{\theta}-t^{2\theta}u^{\theta}.\end{aligned}$$

Using this we confirm, see [C] on page 236, that the group law for U^F is as follows:

$$x_S(t_1,u_1,v_1)x_S(t_2,u_2,v_2) = x_S(t_1+t_2, u_1+u_2-t_1t_2^{3\theta}, v_1+v_2-t_1^2t_2^{3\theta}-t_2u_1+t_1t_2^{3\theta+1}).$$

Replacing positive roots by negative roots and proceeding as above we get that $U^{-F} = \{x'_S(t,u,v) \mid t,u,v \in F_q\}$, where

$$x'_S(t,u,v) = \begin{pmatrix} 1 & . & . & . & . & . & . \\ t^\theta & 1 & . & . & . & . & . \\ t^{\theta+1}-u^\theta & t & 1 & . & . & . & . \\ (tu)^\theta+v^\theta & -u^\theta & -t^\theta & 1 & . & . & . \\ g_1(t,u,v) & (tu)^\theta-v^\theta & -t^{2\theta} & -t^\theta & 1 & . & . \\ g_2(t,u,v) & g_3(t,u,v) & t^{2\theta+1}+v^\theta & t^{\theta+1}-u^\theta & -t & 1 & . \\ g_4(t,u,v) & g_5(t,u,v) & g_6(t,u,v) & (tu)^\theta+v^\theta & u^\theta & -t^\theta & 1 \end{pmatrix}$$

with

$$\begin{aligned} \theta &= 3^m, \\ g_1(t,u,v) &= t^{3\theta+1}+t^{2\theta}u^\theta-(tv)^\theta-u, \\ g_2(t,u,v) &= t^{3\theta+2}+t^{\theta+1}v^\theta-u^\theta t^{2\theta+1}-(uv)^\theta+tu-v, \\ g_3(t,u,v) &= -t^{\theta+1}u^\theta-u^{2\theta}+tv^\theta \\ g_4(t,u,v) &= t^{4\theta+2}+u^\theta t^{3\theta+1}+(tuv)^\theta-v^{2\theta}-u^{\theta+1}+t^\theta v, \\ g_5(t,u,v) &= t^{3\theta+2}+t^\theta u^{2\theta}-(uv)^\theta+v, \\ g_6(t,u,v) &= -t^{3\theta+1}-(tv)^\theta+u. \end{aligned}$$

Here the group law is as follows:

$$x'_S(t_1,u_1,v_1)x'_S(t_2,u_2,v_2) = x'_S(t_1+t_2, u_1+u_2+t_1t_2^{3\theta}, v_1+v_2-t_1^2t_2^{3\theta}+t_2u_1+t_1t_2^{3\theta+1}).$$

We note that the form of the group law in U^F differs from that of U^{-F} by two minus signs. We also remark here that if we let our matrices $X_r(t)$ act on the right, rather than on the left, then the form of the group law for U^F changes to that of U^{-F} and vice versa.

Following [C] Lemma 12.1.1 we define $n_r(t)$ as $X_r(t)X_{-r}(-t^{-1})X_r(t)$ and $h_r(t) = n_r(t)n_r(-1)$. Now every element of T is of the form $h_a(t_1)h_b(t_2)$. Then by Lemma 13.7.1 we have $F(h_a(t_1))F(h_b(t_2)) = h_b(t_1^{3\theta})h_a(t_2^\theta)$. So by Theorem 13.7.4 an element of T is F-invariant iff $t_1 = t_2^\theta$ and $t_2 = t_1^{3\theta}$; i.e. all the $h_a(t)h_b(t^{3\theta})$, where $t \in F_q$, are invariant. Let $t = t_1$, then $T^F = \{h(t) \mid t \in F_q^*\}$, where

$$h(t) = \begin{pmatrix} t^\theta & . & . & . & . & . & . \\ . & t^{1-\theta} & . & . & . & . & . \\ . & . & t^{2\theta-1} & . & . & . & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & t^{1-2\theta} & . & . \\ . & . & . & . & . & t^{\theta-1} & . \\ . & . & . & . & . & . & t^{-\theta} \end{pmatrix}.$$

Finally we note that N^F is generated by T^F and the matrix

$$n := n_{a+b}(1)n_{3a+b}(1) = \begin{pmatrix} . & . & . & . & . & . & -1 \\ . & . & . & . & . & -1 & . \\ . & . & . & . & -1 & . & . \\ . & . & . & -1 & . & . & . \\ . & . & -1 & . & . & . & . \\ . & -1 & . & . & . & . & . \\ -1 & . & . & . & . & . & . \end{pmatrix}.$$

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