Often one has a functor F which one expects to be an equivalence, but without a natural pseudo-inverse. In this case, the following criterion is useful:

Def: Let $F: C \to D$ be a functor.

- F is <u>essentially surjective</u> if

  $\forall d \in D \ \exists c \in C$ and an isomorphism
  $$Fc \to d$$

- F is faithful / full / fully faithful if $\forall c, c' \in C$ the map
  $$\mathrm{Hom}(c,c') \xrightarrow{F} \mathrm{Hom}(Fc, Fc')$$
  is injective / surjective / bijective

$\rightsquigarrow$ **Prop:** A functor F is an equivalence of categories if and only if it is essentially surjective and fully faithful.

<u>Example:</u>

$$\mathrm{Mat}_k \to \mathrm{Vect}_k^{fd}$$
$$n \mapsto k^n$$
$$A \in k^{n \times m} \mapsto \begin{array}{c} k^n \to k^m \\ x \mapsto Ax \end{array}$$

is an equivalence:

- Ess. surj. says that every fd v.s/k is isomorphic to some k^n
- Fully faithful says that $\mathrm{Hom}(k^n, k^m) \cong k^{n \times m}$

<u>Lemma:</u> $F : C \to D$ fully faithful

$c \xrightarrow{f} c'$ in C: f is iso $\Leftrightarrow$ Ff is iso

<u>Pf:</u> "$\Rightarrow$" was an exercise

"$\Leftarrow$": $\exists! g: c' \to c$ s.t. $Fg = (Ff)^{-1}$

$F(g \circ f) = Fg \circ Ff = id_{Fc} = F id_c \Rightarrow g \circ f = id_c$

same way get $f \circ g = id_{c'}$ $\square$

Hence: $F: C \to D$ is an equivalence $\Leftrightarrow$ its fully faithful and induces a bijection between the isomorphism classes of C and D

Ex: . FinSet $\subseteq$ Set full subcat. of fin. sets

. $\mathcal{C}$: $\mathrm{Ob}(\mathcal{C}) = \mathbb{Z}^{\geq 0}$

$n, m \geq 0 \rightsquigarrow \mathrm{Hom}(n,m) = \mathrm{Hom}_{\mathrm{Set}}(\{1,\dots,n\}, \{1,\dots,m\})$

$\rightsquigarrow$ functor $\mathcal{C} \to \mathrm{FinSet}$
$n \mapsto \{1,\dots,n\}$

is an eq. of cat's.

More generally: $\mathcal{C}$ cat.

$S \subseteq \mathrm{Ob}(\mathcal{C})$ s.t. $\forall c \in \mathcal{C}\ \exists c' \in S$ st $c \cong c'$

$\rightsquigarrow \mathcal{C}'$ full subcat of $\mathcal{C}$ with $\mathrm{ob}(\mathcal{C}') = S$

$\rightsquigarrow \mathcal{C}' \to \mathcal{C}$ is an eq.

Ex: Gelfand duality:
X compact Hausdorf top. space

$\rightsquigarrow C(X) = \{X \to \mathbb{C}$ continuous$\}$

Norm: $f \mapsto \|f\| = \max_{x \in X} |f(x)|$

$f \mapsto f^* \in C(X)$: $f^*(x) = \overline{f(x)}$

$\rightsquigarrow$ this is a C^*-alg.

$\rightsquigarrow$ functor $\mathrm{CHTop}^{op} \to C^*\mathrm{Alg}$
$X \mapsto C(X)$

Thm (Gelfand): This is an equivalence.

To prove the prop, we will use the following lemma:

<u>Lemma</u> $\mathcal{C}$ cat. $c \xrightarrow{f} d$, $c \xrightarrow{g} c'$, $d \xrightarrow{h} d'$ isos.

$\leadsto \exists!\ c' \xrightarrow{f'} d'$ s.t. one (or equivalently all) of the following commute:

$$\begin{tikzcd} c \arrow[d, "f"'] & c' \arrow[l, "g^{-1}"'] \arrow[d, "f'"] \\ d \arrow[r, "h"'] & d' \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "g"] \arrow[d, "f"'] & c' \arrow[d, "f'"] \\ d \arrow[r, "h"'] & d' \end{tikzcd} \qquad \begin{tikzcd} c \arrow[d, "f"'] & c' \arrow[l, "g^{-1}"'] \arrow[d, "f'"] \\ d & d' \arrow[l, "h^{-1}"] \end{tikzcd}$$

$$\begin{tikzcd} c \arrow[r, "g"] \arrow[d, "f"'] & c' \arrow[d, "f'"] \\ d & d' \arrow[l, "h^{-1}"] \end{tikzcd}$$

<u>Pf:</u> $f' = h \circ f \circ g^{-1}$ does it.

<u>Lemma:</u> $\mathcal{C}, \mathcal{D}$ cat's

$F, G: \mathcal{C} \to \mathcal{D}$ functors

$\alpha: F \to G$ nat. transf.

α is an iso. of functor

$\iff \forall c \in \mathcal{C}: \alpha_c: Fc \to Gc$ is an iso in $\mathcal{D}$

<u>Pf:</u> "$\Rightarrow$" If $\beta: G \to F$ is an inverse nat. transf, β_c is an inv. to α_c

"$\Leftarrow$", For $c \in \mathcal{C}$ let $\beta_c = \alpha_c^{-1}: Gc \to Fc$.

For $f: c \to d$: $\alpha_d \circ Ff = Ff \circ \alpha_c \Rightarrow Ff \circ \beta_c = \beta_d \circ Ff \Rightarrow \beta$ is nat transf. $\square$

Pf of Prop: first $F: C \to D$ be an equivalence.

Take a pseudo-inverse $G: D \to C$ with isomorphisms $\alpha: F \circ G \xrightarrow{\cong} \mathrm{id}_D$, $\beta: G \circ F \xrightarrow{\cong} \mathrm{id}_C$ of functors.

For $d \in D$: $\alpha_d : FGd \xrightarrow{\cong} d$

$\Rightarrow$ F is essentially surj.

Consider $c \underset{g}{\overset{f}{\rightrightarrows}} c'$ in C with $Ff = Fg$:

$$\begin{tikzcd} GFc \arrow[r, "\beta_c", "\cong"'] \arrow[d, "GFg=GFf"'] & c \arrow[d, "f \text{ or } g"] \\ GFc' \arrow[r, "\cong", "\beta_{c'}"'] & c' \end{tikzcd}$$

Lemma $\Rightarrow f = g$

$\Rightarrow F$ is faithful

$\Rightarrow$ so is G by symmetry

Consider $c, c' \in C$ and $Fc \xrightarrow{h} Fc'$ in D

$$\begin{tikzcd} GFc \arrow[r, "\beta_c", "\cong"'] \arrow[d, "Gh"'] & c \arrow[d, dashed, "f"] \\ GFc' \arrow[r, "\beta_{c'}"'] & c' \end{tikzcd}$$

Lemma $\Rightarrow \exists! \; c \xrightarrow{f} c'$ making this commute

Lemma again $\Rightarrow GFf = Gh$

$\overset{G \text{ faithful}}{\Rightarrow}$ $Ff = h$ $\Rightarrow$ F is full

$\Rightarrow$ F is fully faithful.

Conversely, assume that F is fully faithful + ess. surj.
Via axiom of choice, we choose for each $d \in D$ an object $Gd \in C$ + an isomorphism $FGd \xrightarrow{\alpha_d} d$ in D.
For $d \xrightarrow{e} d'$ in D consider

$$\begin{tikzcd} FGd \arrow[r, "\alpha_d", "\cong"'] \arrow[d, dashed, "\exists!\, n"'] & d \arrow[d, "e"] \\ FGd' \arrow[r, "\alpha_{d'}"'] & d' \end{tikzcd}$$

F fully faithful $\Rightarrow \exists!\ Gd \xrightarrow{Gh} Gd'$ s.t. $FGh = n$

$\rightsquigarrow$ Need to check that we have constructed a functor $G: D \to C$.

- For $d \in D$:

$$\begin{tikzcd} FGd \arrow[r, "\alpha_d"] \arrow[d, "FGid_d / id_{FGd}"'] & d \arrow[d, "id_d"] \\ FGd \arrow[r, "\alpha_d"'] & d \end{tikzcd}$$

Lemma $\Rightarrow FGid_d = id_{FGd} = F\, id_{Gd}$
F faithful $\Rightarrow G id_d = id_{Gd}$.

- $d \xrightarrow{e} d' \xrightarrow{e'} d''$:

$$\begin{tikzcd} FGd \arrow[r, "\alpha_d"] \arrow[d, "F(Ge \circ Ge') \text{ or } FG(e \circ e')"'] & d \arrow[d, "e' \circ e"] \\ FGd'' \arrow[r, "\alpha_{d''}"'] & d'' \end{tikzcd}$$

$\Rightarrow Ge \circ Ge' = G(e \circ e')$
$\Rightarrow G$ is a functor.

By construction of G,

$d \mapsto \alpha_d$ gives a nat transf $\alpha : FG \xrightarrow{\cong} \mathrm{id}_D$

Still need $\beta : GF \xrightarrow{\cong} \mathrm{id}_C$:

For $c \in C$, since F is fully faithful,

$\exists! \ \beta_c : GFc \xrightarrow{\cong} c$ s.t.

$$F\beta_c = \alpha_{Fc} : FGFc \xrightarrow{\cong} Fc$$

For $c \xrightarrow{f} c'$, need

$$\begin{tikzcd}
GFc \arrow[r, "\beta_c"] \arrow[d, "GFf"'] & c \arrow[d, "f"] \\
GFc' \arrow[r, "\beta_{c'}"'] & c'
\end{tikzcd}$$

to commute: Its image under F commutes since α is nat. transf, and hence by faithfulness of F so does this diagram. $\square$