* coproducts: $(c_i)_{i\in I} \in C^I$:
The coproduct of the c_i is an object $\coprod_{i\in I} c_i \in C$ together with morph.
$$\sigma_i : c_i \to \coprod_{i\in I} c_i \quad \text{s.t.}$$
$\forall d \in C, \ (f_i : c_i \to d)_{i\in I} \ \exists f : \coprod_{i\in I} c_i \to d$
s.t. $\forall i$: $f \circ \sigma_i = f_i$

<u>Note:</u> Coproducts in C are the same as products in C^{op}!

<u>Ex:</u> Coproducts exist in:

Set: $\coprod c_i$ = disjoint union of the c_i

Ab: G_i ab grp $\quad \bigoplus_{i\in I} G_i$ direct sum
$$G_j \to \bigoplus_{i\in I} G_i$$
$$g \mapsto (0, \dots, g, \dots 0)$$
$G_i \xrightarrow{h_i} G$ homomorphisms
$$\rightsquigarrow \bigoplus_{i\in I} G_i \xrightarrow{h} G$$
$$(g_i) \mapsto \sum_{i\in I} h_i(g_i)$$
same eg in Mod_R

Grp: free product of groups

Universal properties

Some examples of universal properties

$\mathcal{C}$ cat:

* products: I (index)set, $(c_i)_{i\in I} \in \mathrm{Ob}(\mathcal{C})^I$

  The (direct) product of the c_i is an object $\prod_{i\in I} c_i \in \mathcal{C}$ together with morph.

  $\pi_i : \prod_{i\in I} c_i \to c_i$ for all $i \in I$ s.t.:

  $\forall d \in \mathcal{C}\ \forall (f_i : d \to c_i)_{i\in I}\ \exists!\, f : d \to \prod_{i\in I} c_i$

  s.t. $\forall i \in I: \pi_i \circ f = f_i$

These exist e.g. in Set, Mod_R, Grp, Top (product top!), Cat:

$(\mathcal{C}_i)_{i\in I}$ categories $\rightsquigarrow \prod_{i\in I} \mathcal{C}_i$ cat:

$$\mathrm{Ob}\Big(\prod_{i\in I} \mathcal{C}_i\Big) = \prod_{i\in I} \mathrm{Ob}(\mathcal{C}_i)$$

$$\mathrm{Hom}\big((c_i)_{i\in I}, (c_i')_{i\in I}\big) := \prod_{i\in I} \mathrm{Hom}(c_i, c_i')$$

with termwise composition

* Cat: $(C_i)_{i\in I}$ categories

$\rightsquigarrow \coprod_{i\in I} C_i$ category:

$$Ob\left(\coprod_{i\in I} C_i\right) = \coprod_{i\in I} Ob(C_i)$$

$$c\in C_i,\ c'\in C_j \rightsquigarrow Hom(c,c') = \begin{cases} Hom_{C_i}(c,c') & \text{if } i=j \\ \emptyset & \text{if } i\neq j \end{cases}$$

with the composition from the C_i.

* $c \overset{f}{\underset{g}{\rightrightarrows}} c'$ in C:

an equalizer of f,g is a morphism

$d \xrightarrow{h} c$ s.t. $f\circ h = g\circ h$ and s.t.

$\forall\ e \xrightarrow{k} c$ s.t. $f\circ k = g\circ k$

$\exists!\ e \xrightarrow{\ell} d$
$h\circ \ell = k$

$$\begin{tikzcd} e \arrow[d, dotted, "\ell"'] \arrow[dr, "k"] & & \\ d \arrow[r, "h"'] & c \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & c' \end{tikzcd}$$

Eg: In Set: $d = \{x\in c \mid f(x)=g(x)\} \hookrightarrow c$

In Grp: still $d = \{x\in c \mid f(x)=g(x)\}$
which is now a subgroup of c.

Eg: If $g: c\to c'$ constant hom
$\forall x\in c: x\mapsto 1$

the equalizer of f and g is the kernel of f.

In cat: d = subcat. of c consisting of those objects $x \in c$ for which $f(x) = g(x)$

But as we already saw, it is not a good idea to ask for equality in a category, so this is not very useful. We obtain a more natural construction by replacing "equality" with "isomorphism":

$C \underset{G}{\overset{F}{\rightrightarrows}} C'$ functors $\rightsquigarrow$ cat D:

Objects: (c, h) : $c \in C$

$h : F(c) \to G(c)$ iso in D

morph $(c,h) \to (c', h')$:

morph $c \xrightarrow{g} c'$ in C st.

$$\begin{tikzcd} F(c) \arrow[r, "h"] \arrow[d, "F(g)"'] & G(c) \arrow[d, "G(g)"] \\ F(c') \arrow[r, "h'"'] & G(c') \end{tikzcd}$$

commutes

Have functor: $D \xrightarrow{K} C$

$(c,h) \mapsto c$

$g \mapsto g$

+ an isomorphism of functors

$$F \circ K \overset{\simeq}{=} G \circ K$$

$$(c, h) \mapsto h : F(c) \xrightarrow{\simeq} G(c)$$

$\rightsquigarrow$ This satisfies a univ. prop. analogous to the one of an eq. among all functors $E \overset{L}{\to} C$ together with an isomorphism $F \circ L \simeq G \circ L$

This is the 2-equalizer in the 2-cat. of cat's.

* Dually: A coequalizer of $c \underset{g}{\overset{f}{\rightrightarrows}} c'$ in $\mathcal{C}$ is a morphism $c' \xrightarrow{h} d$ s.t. $h \circ f = h \circ g$ and s.t. $\forall\, c' \xrightarrow{k} e$ s.t. $k \circ f = k \circ g$ $\exists!\, d \xrightarrow{l} e : k = l \circ h$

$$\begin{tikzcd} c \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & c' \arrow[r, "h"] \arrow[dr, "k"'] & d \arrow[d, dashed] \\ & & e \end{tikzcd}$$

Ex: In Set:

Have eq. rel. $\sim$ on c' given by

$\forall\, x, y \in c'$ $\quad x \sim y \iff \exists z \in c : f(z) = x,$
$g(z) = y$

$\rightsquigarrow d = c'/_{\sim}$

So eg if $R \subseteq c' \times c'$ is an eq. rel then $c/_R$ is the coeq. of $R \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} c'$.

In Top: same quotient equipped with quotient topology

In Grp: $c \underset{g}{\overset{f}{\rightrightarrows}} c' \rightsquigarrow N$ = normal subgr.

of c' gen by $\{f(c)g(c)^{-1} \mid c \in C\}$

$\rightsquigarrow d = c'/_N$

So e.g. for $N \trianglelefteq G$ normal, the quotient $G/_N$ is the coequalizer of $N \underset{\text{triv}}{\overset{\text{incl.}}{\rightrightarrows}} G$

One can also form 2-coeq. of categories, but this is more complicated

* Initial object: $c \in C$ is an initial object if $\forall d \in C$ $\exists!$ $c \to d$ in C
(This is the product over the empty set.)

<u>Ex:</u> In Set: $\emptyset$
In Grp: $\{1\}$
In Mod_R: 0
In Cat: $\emptyset$

* Dually: A final object $c \in C$ is one s.t $\forall d \in D$ $\exists!$ $d \to c$ in C

Eg: In Set: $\{*\}$ In Grp: $\{1\}$
In Mod_R: 0 In Cat: $\{*\}$