All small limits can be built out of products and equalizers:

<u>Prop</u>: Let $F: I \to C$ be a small diagram. Then a limit of F is the same as an equalizer of

$$\prod_{i \in I} F_i \underset{b}{\overset{a}{\rightrightarrows}} \prod_{i \xrightarrow{f} j \in I} F_j$$

where a is defined by

$$\prod_{i \in I} F_i \xrightarrow{\pi_i} F_i \xrightarrow{Ff} F_j$$

for $i \xrightarrow{f} j$ and b by

$$\prod_{i \in I} F_i \xrightarrow{\pi_j} F_j .$$

<u>Pf</u> By construction, for $c \in C$, a morphism $c \xrightarrow{g} \prod_{i \in I} F_i$ with

$a \circ g = b \circ g$ is the same as morphisms $C \xrightarrow{g_i} F_i$ s.t. $\forall i \xrightarrow{f} j$ in C the diagram

$$\begin{tikzcd} & F_j \\ C \arrow[ur, "g_j"] \arrow[dr, "g_i"'] & \\ & F_i \arrow[uu, "Ff"'] \end{tikzcd}$$

commutes. So this is the same as a cone $C \to F$. Hence this equalizer has the same UP as $\lim_I F$ and so the claim follows from Yoneda. $\square$

Dually, colimits can be built out of coproducts and coequalizers.

So a category is complete (resp. cocomplete) iff it admits all equal. and all small products (resp. all coeq. and all small coproducts)

This implies:

Ex: Set, Ab, Mod_R, Top are complete and cocomplete.

Some examples:

- $(S, \leq)$ preordered set

$\rightsquigarrow C$: $\mathrm{Ob}(C) = S$

$$\mathrm{Hom}(s,s') = \begin{cases} * & s \leq s' \\ \emptyset & \text{else} \end{cases}$$

(this describes categories in which between any two objects there is at most one morphism.)

$F: I \to C \rightsquigarrow \lim_I F = \inf\{F(i) \mid i \in I\}$

$\mathrm{colim}_I F = \sup\{F(i) \mid i \in I\}$

(if they exist)

So e.g. if $x_1 \geq x_2 \geq \dots$ is a bounded decreasing seq. in $\mathbb{R}$, then $\lim_{i \to \infty} x_i$ is the limit of the diagram $\dots \to x_3 \to x_2 \to x_1$

- limits over the diagram

$$\dots \bullet \to \bullet \to \bullet$$

are also called "inverse limits" "projective" limits.

E.g. $\mathbb{Z}_p = \lim(\dots \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z})$

Colimits over

$$\bullet \to \bullet \to \cdots$$

are called "direct limit" or "inductive limit".

- Fiber products / pullbacks are limits of diagrams

$$\begin{tikzcd} & c_1 \arrow[d, "f"] \\ c_2 \arrow[r, "g"'] & d \end{tikzcd}$$

and are often denoted $c_1 \times_d c_2$.

E.g in set $c_1 \times_d c_2 = \{(x,y) \in c_1 \times c_2 \mid f(x) = g(y)\}$

So e.g if g is an inclusion, $c_1 \times_d c_2 = f^{-1}(c_2) \subset c_1$.

So e.g if $c_2 = \{d\}$ is a single point, then $c_1 \times_d c_2 = f^{-1}(d)$ is the fiber of f over d.

Dually colimits of

$$\begin{tikzcd} d \arrow[r] \arrow[d] & c_1 \\ c_2 & \end{tikzcd}$$

are called pushouts.

<u>Def:</u> Let I be a small category and C, D categories which admit all limits of shape I.
Let $F: I \to C$ be a diagram of shape I and $G: C \to D$ be a functor.
We get a diagram $G \circ F : I \to D$.
Have the univ cone $\lambda : \lim_I F \to F$ given by morph. $\lambda_i : \lim_I F \to F_i$ for all $i \in I$. By applying G we get morph $G(\lambda_i) : G(\lim_I F) \to G(F_i)$, these form a cone $G(\lambda) : G(\lim_I F) \to G \circ F$.
Hence there is a unique morph
$$G(\lim_I F) \to \lim_I G \circ F$$
s.t. $\forall i \in I$ the diagram
$$\begin{tikzcd} G(\lim_I F) \arrow[r] \arrow[dr, "G(\lambda_i)"'] & \lim_I G \circ F \arrow[d, "\lambda_i"] \\ & G(F(i)) \end{tikzcd}$$
$\leftarrow$ univ cone for $G \circ F$

commutes.
We say that G preserves limits of shape I if this morph.
$$G(\lim_I F) \to \lim_I G \circ F$$

is an isomorphism for all F.

Dually, if C and D admit all colimits of shape I, we get a morph.

$$\operatorname{colim}_I G \circ F \to G(\operatorname{colim}_I F),$$

and we say that G preserves colimits of shape I if this is an iso for all F.

<u>Prop</u>: If C and D are (co) cocomplete, then $G: C \to D$ preserves all small (co)limits if and only if it preserves all (co)products and all (co)equalizers.

<u>Pf</u>: This follows from our description of (co)limits in terms of (co)products and (co)equalizers.

$\square$

<u>Ex</u>: • Let $U: Ab \to Set$ be the forgetful functor. This preserves all limits, since it preserves products and equalizers. It does not in general preserve colimits. Eg for coproducts, the map

$$\coprod_{i \in I} G_i \to \bigoplus_{i \in I} G_i$$

is never a bijection.

The same holds for the forgetful functors from Top, Grp, Mod_R, ...

Def: A functor between (co)complete categories which preserves all small (co)limits is called (co)continuous.

Functoriality of (co)limits:

I small cat., C a cat. which admits all limits indexed by I

$\rightsquigarrow F: I \to C \mapsto \lim_I F$ can be extended to a functor $\mathrm{Fun}(I,C) \to C$:

$F, F': I \to C$, $\alpha: F \to F'$ nat. transf.

$\lambda: \lim_I F \to F$, $\lambda': \lim_I F' \to F'$ univ. cones

$\rightsquigarrow \alpha \circ \lambda: \lim_I F \to F'$ cone

$\Rightarrow \exists!\ \lim_I F \xrightarrow[\lim_I \alpha]{} \lim_I F'$ morph.

s.t. $\forall i \in I$ the diagram

$$\begin{tikzcd}
\lim_I F \arrow[r] \arrow[d, "\lambda_i"'] & \lim_I F' \arrow[d, "\lambda'_i"] \\
F_i \arrow[r, "\alpha_i"] & F'_i
\end{tikzcd}$$

commutes.

One checks that this defines a functor

$\lim_I : \mathrm{Fun}(I, C) \to C$

Dually one obtains a functor

$\operatorname{colim}_I : \mathrm{Fun}(I, C) \to C$

if all colimits of shape I exist in C.

Examples:

- $(c_i)_{i\in I}, (c_i')_{i\in I}$ objects of C + morph $(c_i \xrightarrow{\alpha_i} c_i')_{i\in I}$

$\leadsto \prod_{i\in I} c_i \to \prod_{i\in I} c_i'$

-
$$\begin{tikzcd} A_1 \arrow[r, "h_1"] \arrow[d, "f_1"'] & A_2 \arrow[d, "f_2"] \\ B_1 \arrow[r, "h_2"'] & B_2 \end{tikzcd}$$
comm diagram of grps

$\leadsto f_1$ restricts to a hom $\ker(h_1) \to \ker(h_2)$

(since $\ker(h_1) = \mathrm{eq}(A_1 \overset{h_1}{\underset{1}{\rightrightarrows}} A_2)$

Change of index category:

- $I \xrightarrow{H} I'$ functor of small categories
- C cat which admits all I and I'-indexed limits
- $F : I' \to C$ functor $\leadsto F \circ H : I \to C$

$\lambda : \lim_{I'} F \to F$ univ. cone

$i \in I \leadsto \lim_{I'} F \xrightarrow[\lambda_{Hi}]{} FHi$, these give a cone

$\lambda \circ H : \lim_{I'} F \to F \circ H$

$\to \exists!$ morph $\lim_{I'} F \to \lim_I F \circ H$

s.t. $\forall i \in I$ the diagram

$$\begin{array}{ccc} \lim_{I'} F & \to & \lim_I F\circ H \\ \lambda_{Hi} \downarrow & & \downarrow \lambda'_{Hi} \\ FH_i & = & FH_i \end{array}$$

commutes.

As usual, we get a dual result for colimits.

E.g.: $I \subset I'$ sets $\to$ functor $I \to I'$ of discr. cat.

$\to \prod_{i\in I} c_i \to \prod_{i \in I'} c_i'$ nat "proj. morph" in C

(if these products exist in C)