Category Theory Exercise Sheet 9 - Solutions

Exercise 1. For a group G and a functor $F: BG \to Set$, describe $\lim_{BG} F$ and $\operatorname{colim}_{BG} F$.

Solution. By a previous exercise, giving a functor $F: BG \to Set$ amounts to giving a set X = F(*) together with a left G-action on X.

Since $\lim_{BG} F \cong \operatorname{Hom}(\{*\}, \lim_{BG} F) \cong \operatorname{Cone}(\{*\}, F)$, under this identification of functors $BG \to \operatorname{Set}$ with *G*-sets, $\lim_{BG} F$ is the set of *G*-equivariant maps $\{*\} \to X$, where $\{*\}$ is equipped with the trivial *G*-action. Hence $\lim_{BG} F$ is the set of points of *X* which are fixed by *G*.

For a set S, a cone $F \to S$ is the same as a map $X = F(*) \to S$ which is equivariant with respect to the *G*-action on *S* leaving all of *S* fixed. From this it follows that $\operatorname{colim}_{BG} F$ is the quotient of *X* by the smallest equivalence relation \sim for which $g \cdot x \sim x$ for all $g \in G$ and $x \in X$. (This quotient of *X* is called the coinvariants of the *G*-action.)

Exercise 2. For a small category C, show that Fun(C, Set) is complete and cocomplete and describe small (co)limits in this category.

Solution. (Co)limits in Fun(C, Set) can be formed "termwise": Let I be a small category and $F: I \to$ Fun(C, Set) a diagram. For each $c \in C$, there is a functor $ev_c:$ Fun(C, Set) \to Set given by evaluation at c. By composition we obtain a diagram $ev_c \circ F: I \to$ Set whose limit we denoty by L_c . By the functoriality of limits, these objects L_c are naturally functorial in c and so we obtain a functor $L: I \to$ Set together with a cone $F \to L$. The fact that each L_c is a limit of $ev_c \circ F$ implies that L is a limit of F.

The same construction applies to colimits.

Exercise 3. Let C be a locally small category and $c \in C$. Show that the functor $\text{Hom}(c,]: C \to \text{Set}$ preserves all limits which exist in C.

Solution. Let $F: I \to C$ be a diagram which admits a limit in C. By the definition of limits, a morphism $c \to \lim_{I} F$ is the same as a cone $\lambda: c \to F$. But this is the same as morphisms $\lambda_i: c \to Fi$ for all $i \in I$ such that for all $f: i \to j$ in I the morphism λ_j is equal to $Ff \circ \lambda_i$. Hence by the description of limits in Set from the last exercise sheet, the set of such λ is equal to the limit of the functor $\operatorname{Hom}(c, _) \circ F: I \to \operatorname{Set}$. This proves the claim.

Exercise 4. Show that every group can be written as a colimit of a diagram consisting of finitely generated groups.

Lösung. Let G be a group and consider the category I whose objects are the finitely generated subgroups H of G, and whose morphisms are the inclusions of such subgroups. This category comes with a natural inclusion functor $F: I \to \text{Grp}$. The inclusions $H \to G$ induce a homomorphism $\text{colim}_I F \to G$. We claim that this is an isommorphism. Indeed, consider a group G' together with a cone $F \to G'$. Such a cone amounts to giving homomorphisms $H \to G'$ for all finitely generated subgroups H of G which are compatible with inclusions between such subgroups. Since G is the union of all such H, such homomorphisms fit together uniquely to a homomorphism $G \to G'$. This proves the required universal property.